Requested Solutions (§14.6 and §14.7)

(1b) Compute maximum rate of change of $f(x,y,z) = x\ln(yz)$ at point $p = (1, 2, \tfrac{1}{2})$ and the direction in which it occurs.

Sol: Direction of max change is $\nabla f(p)$:

$$\nabla f = \left\langle \ln(yz),\ \frac{x}{yz}\cdot z,\ \frac{x}{yz}\cdot y \right\rangle = \left\langle \ln(yz),\ \frac{x}{y},\ \frac{x}{z} \right\rangle$$

$$\therefore \nabla f(p) = \left\langle \ln(2\cdot\tfrac{1}{2}),\ \frac{1}{2},\ \frac{1}{1/2} \right\rangle = \left\langle \ln(1), \tfrac{1}{2}, 2 \right\rangle = \tfrac{1}{2}\langle 0, 1, 4\rangle$$

The max rate of change is therefore $D_{\vec{u}} f(p)$ where $\vec{u} = \dfrac{\nabla f(p)}{|\nabla f(p)|}$.

$$D_{\vec{u}} f(p) = \vec{u}\cdot\nabla f(p) = \frac{\nabla f(p)}{|\nabla f(p)|}\cdot\nabla f(p) = |\nabla f(p)| = \tfrac{1}{2}\sqrt{0^2+1^2+4^2} = \tfrac{1}{2}\sqrt{17}.$$ ☐

(2d) Compute local minima, maxima, and saddle points of $f(x,y) = xy + e^{-xy}$.

Sol: First find critical points.

Picture of critical points



$$\nabla f = \langle y - ye^{-xy},\ x - xe^{-xy}\rangle = (1-e^{-xy})\langle y, x\rangle$$

$$\nabla f = 0 \iff 1 - e^{-xy} = 0 \quad \text{OR} \quad \langle y, x\rangle = \vec{0}$$

$$\iff e^{-xy} = 1 \quad \text{OR} \quad x = 0 = y$$

$$\iff -xy = 0 \quad \text{OR} \quad x = 0 = y$$

$$\iff (x = 0 \text{ OR } y = 0) \quad \text{OR} \quad x = 0 = y \quad \leftarrow \text{redundant}$$

$$\iff x = 0 \quad \text{OR} \quad y = 0$$

(NB: this means we have infinitely many critical points)

Next compute $D(x,y) = f_{xx}\cdot f_{yy} - f_{xy}^2$ (for the second derivative test).

$$f_{xx} = \frac{\partial}{\partial x}\left[y(1-e^{-xy})\right] = y(-(-y)e^{-xy}) = y^2 e^{-xy}$$

$$f_{xy} = \frac{\partial}{\partial y}\left[y - ye^{-xy}\right] = 1 - (e^{-xy} + y(-x)e^{-xy}) = 1 + (xy-1)e^{-xy}$$

$$f_{yy} = \frac{\partial}{\partial y}\left[x(1-e^{-xy})\right] = x(-(-x)e^{-xy}) = x^2 e^{-xy}$$

$$\therefore D(x,y) = y^2e^{-xy}\cdot x^2e^{-xy} - (1-(xy-1)e^{-xy})^2 = (xy)^2e^{-2xy} - (1-(xy-1)e^{-xy})^2$$

Note that $x=0$ OR $y=0$ implies $xy=0$.

$\therefore$ at all critical points we have $D(x,y) = 0 - (1-(0-1)\cdot 1)^2 = -2^2 = -4 < 0$.

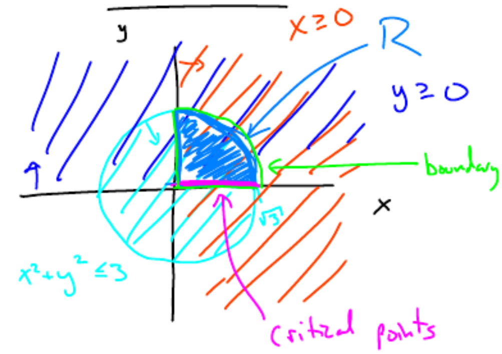
Finally, classify all critical points.

Hence every point of $\{(0,r), (r,0) : r \in \mathbb{R}\}$ is a saddle point. ☐

(3d) Compute absolute extreme values of $f(x,y) = xy^2$ on $R = \{(x,y) : 0 \le x,\ 0 \le y,\ x^2+y^2 \le 3\}$

Sol: First we compute critical points.

PICTURE



$\nabla f = \langle y^2, 2xy \rangle = y \langle y, 2x \rangle$

$\nabla f = \vec{0} \iff y = 0$ OR $(\langle y, 2x \rangle = \vec{0})$

$\iff y = 0$ OR $(x = 0 = y)$ ← Redundant

$\iff y = 0$

NB: Critical points are all contained on the boundary...

Next we parametrize the boundary and compute critical points of the resulting single-variable function $f(\vec{r}(t))$ we do this in 3 pieces.

$\vec{r}_1(t) = \langle t, 0 \rangle$ for $0 \le t \le \sqrt{3}$. ← bottom segment

$\vec{r}_2(t) = \langle 0, t \rangle$ for $0 \le t \le \sqrt{3}$. ← left side segment

$\vec{r}_3(t) = \langle \sqrt{3}\cos(t), \sqrt{3}\sin(t) \rangle$ for $0 \le t \le \frac{\pi}{2}$. ← circular arc.

Note $f(\vec{r}_1(t)) = t \cdot 0^2 = 0$ and $f(\vec{r}_2(t)) = 0 \cdot t^2 = 0$ are constant. Now

$f(\vec{r}_3(t)) = \sqrt{3}\cos(t) \cdot (\sqrt{3}\sin(t))^2 = 3\sqrt{3}\cos(t)\sin^2(t)$; optimizing:

$$f'(\vec{r}_3(t)) = 3\sqrt{3}\left(-\sin(t)\cdot\sin^2(t) + \cos(t)\cdot 2\sin(t)\cos(t)\right)$$
$$= 3\sqrt{3}\sin(t)\left(2\cos^2(t) - \sin^2(t)\right)$$
$$= 3\sqrt{3}\sin(t)\left(2(1-\sin^2(t)) - \sin^2(t)\right)$$
$$= 3\sqrt{3}\sin(t)\left(2 - 3\sin^2(t)\right)$$

$\therefore f'(\vec{r}_3(t)) = 0 \iff \sin(t) = 0$ OR $2 - 3\sin^2(t) = 0$

$\iff \sin(t) = 0$ OR $\sin(t) = \pm\frac{\sqrt{2}}{\sqrt{3}}$

↑ reject $-\frac{\sqrt{2}}{\sqrt{3}}$ because it implies $t \notin [0, \pi/2]$.

$\therefore$ at the critical points we have either

$f(\vec{r}_3(t)) = 3\sqrt{3}\cos(t)\sin^2(t) = 0$

OR $f(\vec{r}_3(t)) = 3\sqrt{3}\cos(t)\sin^2(t) = 3\sqrt{3}\cdot\frac{1}{\sqrt{3}}\cdot\left(\frac{2}{3}\right) = 2$

$\sin(t) = \frac{\sqrt{2}}{\sqrt{3}}$

$\Rightarrow \cos(t) = \frac{1}{\sqrt{3}}$

(triangle: hypotenuse $\sqrt{3}$, opposite $\sqrt{2}$, angle θ, adjacent $\sqrt{3-2} = 1$)

Hence the absolute max is 2 and absolute min is 0 on R. ☐